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Optimal Functional Approximation Using Dynamic Programming

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Introduction

THE problem of approximating some general curve $y = y(x)$, $x_a \leq x \leq x_b$ by some simpler curve $y^*(x)$ is a well-known one For example, Stone[1] considers the case where y^* is a finite number of line segments, i e , $y^* = a_i + b_i x$, $\bar{x}_i \leq x \leq \bar{x}_{i+1}$, $i = 0,\quad , NP$, where $\bar{x}_0 = x_a$, $\bar{x}_{NP+1} = x_b$, and NP is the number of partitions He determines the optimal $y^*(x)$, in the least-squares sense, using the conventional techniques of calculus His solution requires the iterative solution of $(3NP + 2)$ simultaneous linear equations

Bellman and Kotkin[2] obtain a solution to the same problem by the technique of dynamic programming Their solution is much simpler than Stone's, both conceptually and computationally, since the use of the dynamic programming technique enables them to solve for the $(3NP + 2)$ quantities a_i, b_i, and $\bar{x}_i$ in groups of three rather than simultaneously

We wish to consider a simpler problem than the foregoing, namely, the optimal least-squares approximation of an arbitrary function by a step function of a given number of steps In spite of its simplicity, this problem has apparently not been treated elsewhere We give a solution using the technique of dynamic programming This solution is of interest for two reasons First, it provides an introduction to more complicated treatments, such as Ref 2 Also, the solution apparently has several practical applications One of these, the optimal approximation of the theoretical pitch-rate program of a rocket stage, is illustrated below Another possible use is for the optimal evaluation of Riemann-Stieltjes integrals as finite sums The application of this to problems of network synthesis is indicated in Ref 3

Formulation of the Problem

We wish to approximate the arbitrary function $y = y(x)$, $x_a \leq x \leq x_b$, by a step function $y^*(x)$ of the form

$$
\begin{aligned}
y^* &= \alpha_0 && x_a \leq x \leq \bar{x}_1 \\
&= \alpha_k && \bar{x}_k \leq x \leq \bar{x}_{k+1};\ k = 1, \quad , NP - 1 \\
&= \alpha_{NP} && \bar{x}_{NP} \leq x \leq x_b
\end{aligned}
$$

where the constants $\bar{x}_1, \quad , \bar{x}_{NP}, \alpha_0, \alpha_1, \quad , \alpha_{NP}$ are chosen so as to minimize the integral

$$J = \int_{x}^{x_b} (y - y^*)^2 dx$$

Obviously,

$$J = \sum_{k=0}^{NP} \int_{\bar{x}_k}^{\bar{x}_{k+1}} [y(x) - \alpha_k]^2 dx = J\,(\alpha_0, \alpha_1, \quad , \alpha_{NP}, \bar{x}_1, \quad , \bar{x}_{NP})$$

where $\bar{x}_0 = x_a$ and $\bar{x}_{NP+1} = x_b$

Solution

Let us first consider the much simpler problem of choosing the optimal α_k if the optimal $\bar{x}_k$ are given We introduce the function

$$A(x_k, x_{k+1}) = \min_{\alpha_k} \int_{\bar{x}_k}^{\bar{x}_{k+1}} [y(x) - \alpha_k]^2 dx$$

Setting the derivative of the integral with respect to α_k equal to zero, we obtain the optimal value of α_k:

$$\alpha_k = \frac{1}{\bar{x}_{k+1} - \bar{x}_k} \int_{\bar{x}_k}^{\bar{x}_{k+1}} y(x) dx \qquad (k = 0, 1, \quad , NP) \tag{1}$$

Using this value of α_k, we obtain an explicit expression for A:

$$A(\bar{x}_k, \bar{x}_{k+1}) = \int_{\bar{x}_k}^{\bar{x}_{k+1}} [y(x)]^2 dx - \frac{1}{\bar{x}_{k+1} - \bar{x}_k} \left[\int_{\bar{x}_k}^{\bar{x}_{k+1}} y(x) dx \right]^2 \tag{2}$$

Of course, $A(\bar{x}_k, \bar{x}_k) = 0$

We now return to the original problem Let us consider the quantity J, which we will call the minimum "error," to be a function f of the end point of the interval and of the number of partitions, i e , $f_{NP}(x_b) = \min\,[J]$ For the special case of no partitions (one step), $NP = 0$ and

$$f_0(x_b) = A(x_a, x_b)$$

For some arbitrary end point x_i where $x_a \leq x_i \leq x_b$,

$$f_0(x_i) = A(x_a, x_i)$$

Similarly, for the special case of one partition or two steps, we have

$$f_1(x_i) = \min_{x_a \leq x_j \leq x_i} [f_0(x_j) + A(x_j, x_i)]$$

This equation merely states the truism that, in order to minimize J with a single partition, the partition must be placed at the optimal point within the interval However, it indicates a method of solving the problem for $NP > 1$

The principle of optimality[4] states that an optimal policy has the property that, whatever the initial state and initial

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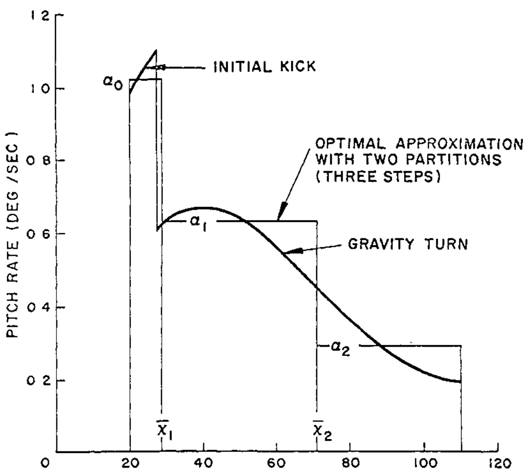

Fig 1 Pitch-rate problem

decision, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision

This means, in our problem, that, wherever the first $(NP - 1)$ partitions have been placed, the NPth partition must be located so as to minimize the sum

$$\sum_{k=0}^{NP-1} \int_{\bar{x}_k}^{\bar{x}_{k+1}} [y(x) - \alpha_k]^2 dx + \int_{\bar{x}_{NP}}^{x_b} [y(x) - \alpha_{NP}]^2 dx$$

We are thus led to the recurrence relations

$$f_k(x_i) = \min_{x_a \le x_j \le x_i} [f_{k-1}(x_j) + A(x_j, x_i)] \qquad (k = NP, NP - 1, \quad , 2, 1)$$

$$f_0(x_i) = A(x_a, x_i)$$

where $A(x_j, x_i)$ is defined in Eq (2) The functions $f_k(x_i)$ are computed for $x_a \le x_i \le x_b$ and $k = 0, 1, 2, \quad , NP$ The values of x_j which minimize the quantities in brackets are also recorded and denoted by $\bar{x}_k(x_i)$, $k = 1, 2, \quad , NP$ Then the optimal partition locations are given by $\bar{x}_{NP} = \bar{x}_{NP}(x_b)$, $\bar{x}_{NP-1} = \bar{x}_{NP-1}(\bar{x}_{NP})$, $\quad , \bar{x}_2 = \bar{x}_2(\bar{x}_3)$, $\bar{x}_1 = \bar{x}_1(\bar{x}_2)$

Once the optimal values of the $\bar{x}_k$ are known, then we can use Eq (1) to find the optimal α_k

Computational Aspects

A computer program that performs the functional approximation described in the previous section is available from the author[5] The program assumes that the function $y(x)$ is available as a discrete set of n values

The computations are such that the desired information (α_k, $k = 0, \quad , NP$; $\bar{x}_k$, $k = 1, \quad , NP$) is readily available for all NP less than the desired value, and all of these results are provided by the program The errors, i e , the values of $f_{NP}(x_b)$ are also provided

All integrations are performed using the trapezoid rule, i e , $y(x)$ is assumed to be linear between the discrete points at which it is given A less simple, more accurate, quadrature formula could be used if additional accuracy were required for some application

It should also be noted that the optimal partition locations must be chosen from among the n discrete points at which $y(x)$ is specified The errors introduced by this approximation remain negligible as long as n is relatively large compared to NP The actual magnitude of these errors can be examined by referring to simple problems for which an exact solution is available[5]

Example

A typical desired pitch-rate program for a rocket stage is shown in Fig 1 The function is discontinuous, consisting of an initial kick followed by a gravity turn For technological reasons, it is often necessary to follow a series of steps rather than attempting to follow the theoretical curve exactly Then it is of interest to have the step curve as close as possible to the theoretical curve

This particular problem (for $NP \le 50$ and $n = 91$) required less than 2 min on the IBM 7094 computer The solution for $NP = 2$ is drawn on Fig 1 for comparison with the theoretical curve

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Flutter of Two Parallel Flat Plates Connected by an Elastic Medium

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Nomenclatuı e

$A_\pm$	=	$R_{x_\pm} - 2(a/b)^2$
a,b	=	plate length and width, see Fig 1
$B_\pm$	=	$(\Omega_\pm^2/\pi^4) + (a/b)^2 R_{y_\pm} - (a/b)^4$
$D_\pm$	=	plate flexural stiffness
$h_\pm$	=	thickness of plate
k	=	elastic spring constant
l	=	lateral aerodynamic load
M	=	Mach number
$N_{x_\pm}$, $N_{y_\pm}$	=	midplane force intensities, positive in compression
q	=	dynamic pıessure, $\rho U^2/2$
$R_{x_\pm}$	=	$a^2 N_{x\pm}/\pi^2 D_\pm$
$R_{y_\pm}$	=	$a^2 N_{y\pm}/\pi^2 D_\pm$
$S_\pm$	=	spring stiffness parameter, $ka^4/\pi^4 D_\pm$
t	=	time
U	=	freestream velocity
$w_\pm$	=	lateral deflection of plate
x,y	=	Cartesian coordinates, see Fig 1
β	=	$(M^2 - 1)^{1/2}$
$\gamma_\pm$	=	mass density of plate
λ	=	$2qa^3/\beta D_+$
ρ	=	mass density of air
ω	=	circular frequenc)
$\Omega_\pm^2$	=	$\omega^2 a^4 \gamma_\pm h_\pm / D_\pm$
$+ \; -$	=	subscripts refer to upper and lower plate, respectively

Introduction

THE flutter behavior of a structural configuration consisting of two rectangular, simply supported, parallel plates laterally connected by many closely spaced linear springs is investigated The configuration analyzed is shown in Fig 1 The upper plate has air flowing at supersonic speed over the

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